Bipancyclicity in k-ary n-cubes with faulty edges under a conditional fault assumption

Yonghong Xiang and Iain A. Stewart
School of Engineering and Computing Sciences
Durham University
Science Labs, South Road
Durham DH1 3LE, U.K.

Abstract

We prove that a k-ary 2-cube Q_2^k with 3 faulty edges but where every vertex is incident with at least 2 healthy edges is bipancyclic, if $k \geq 3$, and k-pancyclic, if $k \geq 5$ is odd (these results are optimal). We go on to show that when $k \geq 4$ is even and $n \geq 3$, any k-ary n-cube Q_n^k with at most $4n - 5$ faulty edges so that every vertex is incident with at least 2 healthy edges is bipancyclic, and that this result is optimal.

Keywords: Interconnection networks. k-ary n-cubes. Fault-tolerance. Bipancyclicity.

1 Introduction

Low-dimensional tori are regularly used as interconnection networks in distributed-memory parallel computers. For example, the Alpha 21364-based HP GS1280 machine [10], the iWarp [6] and the Cray X1E vector computer have a two-dimensional torus as their interconnection networks, while the Cray T3D and T3E [22] have three-dimensional tori as theirs. Furthermore, two-dimensional mesh and torus topologies are popular choices for networks-on-chips [28]. This has helped to motivate a considerable amount of work on the structural aspects of (arbitrary dimensional) tori, and in particular their uniform variants k-ary n-cubes, that are relevant to parallel computing as well as being of interest in purely graph-theoretic terms. For example, the k-ary n-cube Q_n^k has the following basic properties: it is vertex- and edge-symmetric [1]; it is Hamiltonian [3, 7]; it has diameter $n\lfloor \frac{k}{2} \rfloor$ [3, 7]; it has a recursive decomposition;

and it contains embeddings of many important interconnection networks such as cycles (of certain lengths) [1], meshes (of certain dimensions) [3] and even hypercubes (of certain dimensions) [7]. Moreover, it has admirable properties in relation to routing, broadcasting and communication in general (see, for example, [1, 7, 11]).

Of particular relevance to us are some recent results concerning paths and cycles embedded within k-ary n-cubes. Paths and cycles are fundamental in parallel computing; for not only is there a multitude of algorithms specifically designed for linear arrays of processors and cycles of processors but paths and cycles appear as data structures in many more algorithms for parallel machines whose processors are interconnected in a wide variety of topologies. We shall be concerned with questions relating to Hamiltonicity, pancyclicity, bipancyclicity and edge-bipancyclicity (these concepts are defined in the next section). The existence of these properties in an interconnection network enables a much higher degree of flexibility with regard to the simulation of linear arrays or cycles of processors. The results of [24] are of significance to us, where earlier results due to Hsieh, Lin and Huang [16] and to Wang, An, Pan, Wang and Qu [27] were extended and the situation as regards the pancyclicity and bipancyclicity of Q_n^k was settled. Amongst other results, it was shown in [24] that Q_n^k is edge-bipancyclic, when $n \geq 2$ and $k \geq 3$, and k-pancyclic, when $n \geq 2$ and $k \geq 3$ is odd.

As more and more processors are incorporated into parallel machines, faults become more common, be it faults in the processors or on the connections between processors. Of course, the temporary unavailability of a connection between two processors due to, for example, high traffic can also be regarded as a fault. Given the significant cost of parallel machines, we would prefer to be able to tolerate (small numbers of) faults and still be able to use our parallel machine. Whilst 'static' structural results such as those mentioned above are important, we are interested here in the tolerance of k-ary n-cubes when a (limited) number of edges are faulty (that is, are missing). In particular, we are interested in how many faulty edges a k-ary n-cube Q_n^k can tolerate yet still remain bipancyclic and k-pancyclic.

As the k-ary n-cube Q_n^k has degree $2n$, an immediate upper bound on the number of faulty edges Q_n^k can tolerate and still remain bipancyclic or k-pancyclic is clearly $2n-2$ (for we can make the edges from a vertex to $2n-1$ of its neighbours faulty and there clearly can be no cycle through the vertex). Consequently, many studies assume the conditional fault assumption on the distribution of the faults so that no matter how many faulty edges there are, it is always the case that every vertex is incident with at least 2 healthy edges (the legitimacy of this conditional fault assumption is given credence as there is a very small probability that a configuration of faulty edges will be such as to make a vertex of one of our networks have degree less than 2). For example, under this conditional fault assumption: it was shown in [2] that Q_n^k with $4n-5$ faulty edges still has a Hamiltonian cycle (and that this result is optimal); it was shown in [26] that an n-dimensional alternating group graph with $4n-13$ faulty edges still has a Hamiltonian cycle (and that this result is optimal); it was shown in [19] that an n-dimensional crossed cube with $2n-5$ faulty edges still has a Hamiltonian cycle (and that this result is optimal); and in [15] a more general consideration of matching composition networks was made with regard to whether they remain Hamiltonian under a limited number of faults. Other Hamiltonicity results under our conditional fault assumption are available in, for example, [8, 13, 14, 17, 18, 25]. As far as we are aware, [25] is the only paper to have considered pancyclicity issues in a family of interconnection networks in the presence of faulty edges and under our conditional fault assumption: in [25] it was proven that, under our conditional fault assumption, an n-dimensional hypercube with $2n-5$ faulty edges remains bipancyclic (and that this result is optimal).

In this paper we resolve the situation as regards pancyclicity and bipancyclicity in k-ary 2-cubes under our conditional fault assumption. In particular, we prove that a k-ary 2-cube Q_2^k with 3 faulty edges but where every vertex is incident with at least 2 healthy edges is bipancyclic, if $k \geq 3$, and k-pancyclic, if $k \geq 5$ is odd (these results are optimal). We go on to show that when $k \geq 4$ is even and $n \geq 3$, any k-ary n-cube Q_n^k with at most $4n-5$ faulty edges so that every vertex is incident with at least 2 healthy edges is bipancyclic, and that this result is optimal. The proof of this latter result is long and complicated and uses a variety of techniques concerning the combinatorial manipulation of k-ary n-cubes (in the presence of faults) that are interesting in their own right. In the next section we detail the basic definitions relevant to this paper, and in Section 3 we prove our results for the k-ary 2-cube. In Section 4 we prove our main result concerning the bipancyclicity of Q_n^k when k is even. We give our conclusions and directions for further research in Section 5.

2 Basic definitions

For $k \geq 3$ and $n \geq 1$, a *k-ary n-cube* Q_n^k has a vertex set of $\{0, 1, \ldots, k-1\}^n$ and there is an edge $((u_n, u_{n-1}, \ldots, u_1), (v_n, v_{n-1}, \ldots, v_1))$ if, and only if, $|u_i - v_i| = 1$ (mod k), for some $i \in \{1, 2, \ldots, n\}$, with $u_j = v_j$, for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$; such an edge is termed as lying in *dimension* i (throughout, arithmetic on the components of vertices is modulo k). A $k_1 \times k_2$ *torus* has vertex set $\{(u, v) : u \in \{0, 1, \ldots, k_1 -$

$1\}, v \in \{0, 1, \ldots, k_2 - 1\}\}$ and there is an edge $((u_1, u_2), (v_1, v_2))$ if, and only if, $|u_i - v_i| = 1$ (mod k_i), for some $i \in \{1, 2\}$, with $u_j = v_j$, for $j \neq i$. We sometimes refer to edges of the form $((i, j), (i, j + 1))$ (resp. $((i, j), (i + 1, j))$) in a k-ary 2-cube or a $k_1 \times k_2$ torus as lying on *row* i (resp. in *column* j).

Let $i \in \{1, 2, \ldots, n\}$. We say that we *partition* Q_n^k *over dimension* i if we consider Q_n^k to be the disjoint union of copies $Q_0, Q_1, \ldots, Q_{k-1}$ of Q_{n-1}^k as follows: for each $j \in \{0, 1, \ldots, k-1\}$, Q_j is the subgraph of Q_n^k induced by the vertices of Q_n^k whose ith component is j (we suppress i, n and k in the notation as they are always understood). Clearly, all edges not in some Q_j lie in dimension i. Suppose that we have partitioned Q_n^k over dimension i as $Q_0, Q_1, \ldots, Q_{k-1}$ and $x = (x_n, x_{n-1}, \ldots, x_1)$ is some vertex of some Q_j. The vertex $(x_n, \ldots, x_{i+1}, m, x_{i-1}, \ldots, x_1)$ of Q_m is denoted as $n_m(x)$ (and so $x = n_j(x)$).

We consider k-ary n-cubes with *faulty edges* (or simply *faults*); that is, where certain edges are missing. Thus, a *faulty* k-ary n-cube is really just a copy of Q_n^k where some edges, the faulty edges, are missing, and where we refer to the edges that remain as the *healthy* edges. Even though our faulty edges are regarded as missing edges, we still say, for example, that a vertex v is incident with some faulty edge e when the edge e was originally incident with v before it was removed. On occasion, we want to emphasise that all edges of some sub-graph are healthy and so we say, for example, that a cycle or a path is healthy.

A *conditional fault assumption* is an assumption relating to the faults (in our case, faulty edges) and their distribution within an interconnection network (which for us is always a k-ary n-cube). The conditional fault assumption we make is that the distribution of faults is such that no vertex in any faulty k-ary n-cube is ever incident with less than 2 healthy edges (that is, has degree less than 2 when we regard our faulty k-ary n-cube as being a k-ary n-cube with some edges missing).

A graph on n vertices is: *pancyclic* if it contains a cycle of every length from 3 up to n; *edge-pancyclic* if there is cycle of every length from 3 up to n containing any given edge; and m-*pancyclic* if it contains a cycle of every length from m up to n. Of course, no bipartite graph can be pancyclic (as there can be no odd length cycles); consequently, a notion of pancyclicity has been devised for bipartite graphs. A bipartite graph on n vertices is *bipancyclic* if there is an even length cycle of every even length from 4 up to n, and *edge-bipancyclic* if there is an even length cycle of every even length from 4 up to n containing any given edge. Even though the notions of bipancyclicity and edge-bipancyclicity have been devised to primarily apply to bipartite graphs, it still makes sense to apply them to non-bipartite graphs too. We shall be building cycles of various lengths in faulty k-ary n-cubes. We say that a cycle C, of length c, say, can be *progressively shortened* to a cycle of length c', say, if starting from C we can iteratively apply the following construction to obtain cycles of all lengths c, $c-2$, $c-4$, down to c': in the current cycle C', replace a sub-path (u, v, w, y) of length 3 with the edge (u, y) to obtain a cycle of length 2 less than the length of C' (note that we also describe a cycle in a graph as a sequence of vertices so that consecutive vertices in the sequence are joined by an edge in the cycle, as well as there being an edge from the last vertex of the sequence to the first).

If π is a property of graphs then a graph G is said to be *m-edge-fault-tolerant* π if G still has property π even after the removal of at most m edges from G. Thus, for example, to say that a k-ary n-cube Q_n^k is $(4n-5)$-edge-fault-tolerant bipancyclic under the conditional fault assumption that no vertex is incident with less than 2 healthy edges means that no matter which $4n-5$ edges we remove from Q_n^k, so long as no vertex in the resulting graph has degree less than 2, there is a cycle of every even length m where $4 \leq m \leq k^n$.

A graph G is *vertex-symmetric* if given any two distinct vertices u and v of G, there is an automorphism of G mapping u to v. Similarly, a graph is *edge-symmetric* if given two distinct edges e and f of G (possibly incident), there is an automorphism of G mapping e to f. We shall use the fact that Q_n^k is edge-symmetric [2] throughout. The k-ary 2-cube Q_2^k has a number of automorphisms. For example, the maps $(i, j) \mapsto (i+1, j)$, $(i, j) \mapsto (i, j+1)$, $(i, j) \mapsto (k-1-i, j)$, and $(i, j) \mapsto (i, k-1-j)$ are all automorphisms of Q_2^k.

3 The case when $n = 2$

We begin by dealing with the k-ary 2-cube Q_2^k in which there are 3 faulty edges.

Lemma 1 *Consider a k-ary 2-cube Q_2^k, for some even $k \geq 6$, in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq k^2$.*

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2; that is, these 2 faulty edges are column edges. As Q_2^k is edge-symmetric [1], w.l.o.g. we may assume that the edge $((0, 0), (k-1, 0))$ is faulty.

<u>Case 1</u>: all faulty edges lie in dimension 2; that is, all faults are column faults.

We remark that throughout the proof of this case, we never use edges of the form $((k-1, i), (0, i))$ and so we may simply ignore the faulty edge $((0, 0), (k-1, 0))$ and assume that we are working in the $k \times k$ grid with wrap-around edges of the form $((i, k-1), (i, 0))$, for $i = 0, 1, \ldots, k-1$. Consider the Hamiltonian cycle of Q_2^k as pictured in Fig. 1(*a*) (although we have only drawn a Hamiltonian cycle in Q_2^8, the analogous cycle in Q_2^k, for any even $k \geq 6$, should be clear). This cycle, which we call the E-cycle rooted at $(0, 0)$, can be translated (horizontally) by simply increasing the first component of every vertex by 2, and yet another cycle can be obtained by increasing the first component of every vertex by 4. Note that the 3 Hamiltonian cycles so obtained are edge-disjoint when we consider only column edges (as $k \geq 6$). Thus, at least one of these Hamiltonian cycles contains only healthy edges; call it C. W.l.o.g. we may assume that C is the E-cycle rooted at $(0, 0)$.

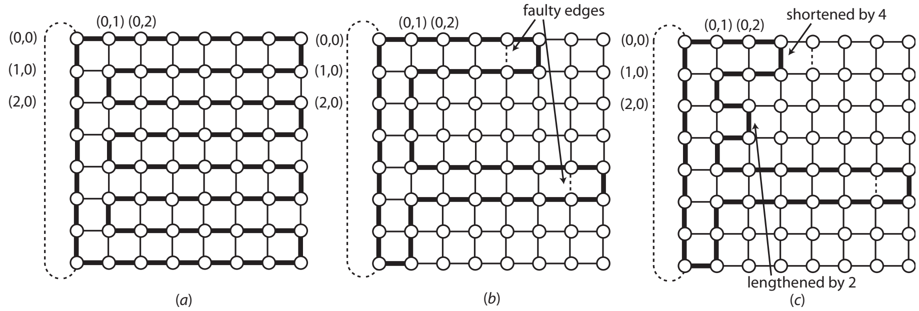

Figure 1. An E-cycle and progressive shortening in Q_2^8.

If our copy of Q_2^k had no faulty edges then we could clearly progressively shorten C by at each step removing 3 edges and including 1 new edge, so that we obtain a healthy cycle of every even length l for which $4 \leq l \leq k^2$. However, in the process of progressively shortening our cycle, we might try to include a new edge that is actually faulty (note that we encounter at most 2 faulty edges in our process of progressive shortening). We deal with this situation as follows. Our process of progressive shortening begins by introducing column edges; so, with reference to Fig. 1(*a*), we shorten our cycle 'from the right-hand side' (note that there are $\frac{k}{2} \geq 3$ ways in which we could do this). We also ensure that we shorten the cycle in this way as much as we can before having to deal with attempting to introduce a faulty edge. If we try to introduce a faulty edge then we simply 'jump' that particular iteration of our process of progressive shortening and instead of shortening the cycle by 2, we shorten the cycle by 4, unless the next edge to be introduced is faulty too, when we shorten the cycle by 6. Note that because of how we have chosen to progressively shorten our cycle up until this point, we can simultaneously lengthen our cycle by 2 or 4 (in a different part of the cycle) to ensure that we obtain cycles of all the required lengths. The process can be visualized in Fig. 1 where we have encountered a faulty edge in Fig. 1(*b*) and 'jumped' over it in

Fig. 1(c) as well as lengthened our cycle by 2 (there are subtleties when at least 1 of our column faults is of the form $((i,1),(i+1,1))$ but such configurations can easily be coped with).

Case 2: exactly 2 faulty edges lie in dimension 2; that is, there are exactly 2 column faults.

Case 2.1: The 3 faults are such that they do not form a path of length 3 starting with a column edge, followed by a row edge, and ending with a column edge.

W.l.o.g. we may assume that $((0,0),(k-1,0))$ is a faulty edge and that the row fault is not incident with any column fault different from $((0,0),(k-1,0))$. As in Case 1, we never use edges of the form $((k-1,i),(0,i))$ and so we may simply ignore the faulty edge $((0,0),(k-1,0))$ and assume that we are working in the $k \times k$ grid with wrap-around edges of the form $((i,k-1),(i,0))$, for $i = 0,1,\ldots,k-1$. By translating the E-cycle rooted at $(0,0)$ (as illustrated in Fig. 1(a)) by increasing the first component of every vertex by 1, 2, and so on, w.l.o.g. we may assume that: we have a (not necessarily healthy) Hamiltonian cycle C in Q_2^k that is the E-cycle rooted at $(0,0)$; C contains no faulty column edge; and C contains a faulty row edge and this faulty edge is one of $\{((i,k-4),(i,k-3)),((i,k-3),(i,k-2)),((i,k-2),(i,k-1))\}$, for some $i \in \{0,1,\ldots,k-1\}$ (note that if C contains no faulty row edge then we can progressively shorten C, as in Case 1, so as to obtain healthy cycles of every even length l where l is such that $4 \leq l \leq k^2$, 'jumping' over faults as in Case 1).

Depending upon where the faulty row edge lies, we now amend our cycle C analogously to the illustration in Fig. 2(a) where: we remove the faulty row edge and its 'opposite' (healthy) edge on the same 'branch' of the E-cycle; we include the column edges which join the two edges just removed; and we 'join' the resulting disconnected path to the main cycle by removing a column edge and including two row edges. What results is a healthy Hamiltonian cycle C that can be progressively shortened just as we did in Case 1, above (and 'jumping' over the faulty column edge, should it be encountered), so that we obtain a cycle of length l for every even l for which $4 \leq l \leq k^2$.

Case 2.2: the 3 faults form a path in the form of a column edge followed by a row edge followed by a column edge.

W.l.o.g. the faults are $((k-1,0),(0,0))$, $((0,0),(0,1))$ and $((0,1),(k-1,1))$ or the faults are $((k-1,0),(0,0))$, $((0,0),(0,1))$ and $((0,1),(1,1))$. In the first case, we have a healthy E-cycle rooted at $(0,1)$ and so the result clearly follows (by proceeding as in Case 1). In the second case, the Hamiltonian cycle as depicted in Fig. 2(b) can be progressively shortened so that we obtain a healthy cycle of length l for every even l for which $2k \leq l \leq k^2$ (although we have only depicted this Hamiltonian cycle in Q_2^8, the analogous cycle in Q_2^k, for any even $k \geq 6$, should be clear). It is trivial to obtain healthy cycles of even length from 4 up to $2k-2$. The result follows. $\square$

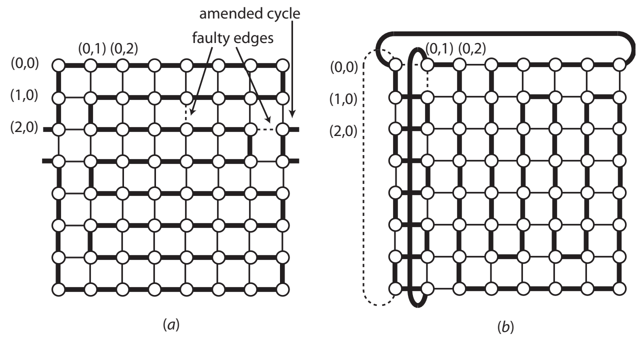

Figure 2. Amending the E-cycle in Q_2^8 and a Hamiltonian cycle.

Lemma 2 *Consider a 4-ary 2-cube Q_2^4 in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq 16$.*

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case 1: all faulty edges are column edges.

For $i = 0, 1, 2, 3$, let C_i be the cycle $((i,0),(i,1),(i,2),(i,3))$. Both edges of at least one of the edge-pairs

- $\{((0,0),(1,0)),((0,1),(1,1))\}$
- $\{((0,0),(3,0)),((0,1),(3,1))\}$
- $\{((0,2),(1,2)),((0,3),(1,3))\}$
- $\{((0,2),(3,2)),((0,3),(3,3))\}$

are healthy; thus, we can 'join' C_0 to C_1 or C_3, as appropriate and using these edges, to obtain a healthy cycle of length 8. We can also join C_0 to an edge of C_1 or C_3, as appropriate and using these edges, to obtain a healthy cycle of length 6 (see Fig. 3(*a*) and Fig. 3(*b*) for an illustration of these constructions). By continuing in the same way and using the same reasoning with the resulting cycle of length 8, we can ultimately obtain healthy cycles of every even length from 4 up to 16 as required.

Case 2: exactly 2 faulty edges are column edges.

W.l.o.g. we may assume that the edge $((0,0),(0,1))$ is faulty. If it is the case that either $((0,0),(1,0))$ and $((0,1),(1,1))$ are both healthy or $((0,0),(3,0))$ and $((0,1),(3,1))$ are both healthy then we may proceed as we did in Case 1 and iteratively obtain healthy cycles of all the required lengths. Thus, suppose that 1 of the 2 edges $((0,0),(1,0))$ and $((0,1),(1,1))$ is faulty and 1 of the 2 edges $((0,0),(3,0))$ and $((0,1),(3,1))$ is faulty. W.l.o.g. we may assume that the 3 faulty edges are $((0,0),(0,1))$, $((0,0),(1,0))$, and $((0,1),(3,1))$ or they are $((0,0),(0,1))$, $((0,0),(3,0))$, and $((0,1),(3,1))$ (recall our conditional fault assumption). In the former case, the Hamiltonian cycle in Fig. 3(c) can clearly be progressively shortened so that we obtain healthy cycles of lengths 14, 12, 10, and 8, and it is trivial to find healthy cycles of lengths 6 and 4. In the latter case, we have a healthy E-cycle rooted at $(0,1)$ and so can proceed as we did in Case 1 of Lemma 1. The result follows. □

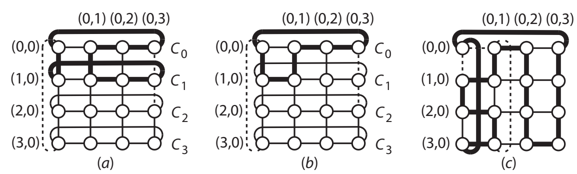

Figure 3. Joining cycles and a Hamiltonian cycle in a faulty Q_2^4.

Lemma 3 *Consider a k-ary 2-cube Q_2^k, for some odd $k \geq 7$, in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq k^2 - 1$.*

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case 1: all faults are column edges.

Let G be the wrap-around grid induced by the vertices of $\{(i,j) : 0 \leq i \leq k-2, 0 \leq j \leq k-1\}$. W.l.o.g. we may assume that $((0,0),(k-1,0))$ is a faulty edge and that at least 1 other fault lies in G. The constructions of Case 1 of Lemma 1 apply to G and suffice for us to build a healthy cycle of length l for every even l such that $4 \leq l \leq (k-1)k$. Moreover, the cycle C of length $(k-1)k$ spanning G is such that it contains a sub-path P of length $k-1$ consisting of all the vertices of row $k-2$. If G does not contain a faulty edge joining a vertex in row $k-2$ to a vertex in row $k-1$ then we can easily obtain a healthy cycle of every even length l where l is such that $(k-1)k \leq l \leq k^2-1$ (by replacing alternating edges $((k-2,j)(k-2,j+1))$ of P with paths $(((k-2,j),(k-1,j)),((k-1,j),(k-1,j+1)),((k-1,j+1),(k-2,j+1)))$. So, suppose that $((k-2,j),(k-1,j))$ is a faulty edge. Thus, G contains only 1 fault.

By translating C if necessary, w.l.o.g. we may assume that $(k-2, j)$ is the rth vertex on P, for some odd r. Hence, by proceeding similarly we can obtain a healthy cycle of every even length l where l is such that $(k-1)k \leq l \leq k^2 - 1$ (the construction is depicted in Fig. 4(a)).

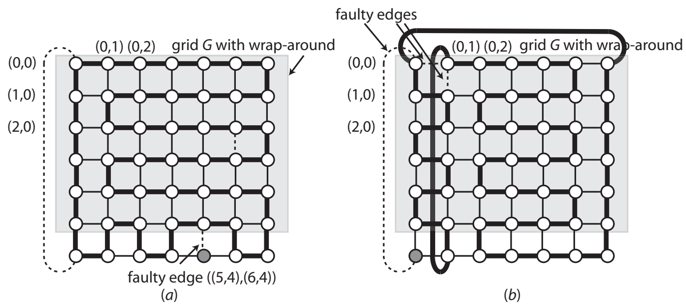

Figure 4. Cycles in Q_2^7.

Case 2: exactly 2 faults are column edges.

Case 2.1: The 3 faults do not form a path of length 3 starting with a column edge, followed by a row edge and ending with a column edge.

W.l.o.g. we may assume that: $((0,0),(k-1,0))$ is a faulty edge; the row fault $((i,j),(i,j+1))$ is such that $i \leq \frac{k-1}{2}$; and the row fault is not incident with any column fault except possibly $((0,0),(k-1,0))$. Let G be the wrap-around grid induced by the vertices of $\{(i,j) : 0 \leq i \leq k-2, 0 \leq j \leq k-1\}$. The constructions of Case 2.1 of Lemma 1 apply to G and suffice for us to build in G a healthy cycle of length l for every even l such that $4 \leq l \leq (k-1)k$. The cycle C of length $(k-1)k$ so constructed is such that it contains a sub-path P of length $k-1$ consisting of all the vertices of row $k-2$. If there does not exist a faulty edge joining a vertex in row $k-2$ to a vertex in row $k-1$ then we can easily obtain a healthy cycle of every even length l where l is such that $(k-1)k \leq l \leq k^2-1$ (just as we did above). If there is a fault $((k-2,j),(k-1,j))$ then we ensure that when we construct our healthy cycle C of length $(k-1)k$ above, the vertex $(k-2,j)$ is the rth vertex on P, for some odd r. Consequently, we can obtain a healthy cycle of every even length l where l is such that $(k-1)k \leq l \leq k^2-1$.

Case 2.2: the 3 faults form a path in the form of a column edge followed by a row edge followed by a column edge.

W.l.o.g. the faults are $((k-1,0),(0,0))$, $((0,0),(0,1))$ and $((0,1),(k-1,1))$ or the faults are $((k-1,0),(0,0))$, $((0,0),(0,1))$ and $((0,1),(1,1))$. In the first case, we have a healthy E-cycle rooted at $(0,1)$ and so the result clearly follows (using the above arguments). In the second case, the cycle of length $k^2 - 1$ as depicted in Fig. 4(*b*) can be progressively shortened so that we obtain a healthy cycle of length l for every even l for which $2k \leq l \leq k^2 - 1$ (although we have only depicted this cycle in Q_2^7, the analogous cycle in Q_2^k, for any odd $k \geq 7$, should be clear). It is trivial to obtain healthy cycles of every even length from 4 up to $2k - 2$. The result follows. $\square$

Lemma 4 *Consider a 5-ary 2-cube Q_2^5 in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every even l such that $4 \leq l \leq 24$.*

Proof There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case 1: all faulty edges are column edges.

W.l.o.g. we may assume that the edge $((0,0),(4,0))$ is faulty and that no other faulty edge joins a vertex in row 0 and a vertex in row $k-1$ (otherwise obtaining the result is trivial: simply assume we are working in the fault-free 5×5 grid with wrap-around edges of the form $((i,4),(i,0))$, find a cycle of an appropriate length and then, if necessary, translate, via an appropriate automorphism, so that the column fault does not lie on the translated cycle).

Suppose that 2 faulty edges are of the form $((i,j),(i+1,j))$, for some fixed $i \in \{0,1,2,3\}$. W.l.o.g. $i = 0$ or $i = 1$. It is trivial to see that if $i = 0$ then in the subgraph of Q_2^5 induced by the vertices of $\{(0,i),(1,i) : 0 \leq i \leq 4\}$ there are healthy cycles of lengths 4, 6, 8 and 10. Moreover, w.l.o.g. we may assume that the cycle of length 10 is $((0,0),(0,1),\ldots,(0,4),(1,4),(1,3),\ldots,(1,0))$. Of course, an analogous statement can be made if there are 2 faulty edges of the form $((1,j),(2,j))$. Regardless, it is trivial to extend any such cycle of length 10 so as to obtain healthy cycles of all even lengths l where $4 \leq l \leq 24$. These extensions can be visualized as in Fig. 5 where a cycle of length 24 is shown that can be progressively shortened so as to obtain healthy cycles of any even length l for which $10 \leq l \leq 24$ (Fig. 5(*a*) corresponds to the case when $i = 0$ and Fig. 5(*b*) to that when $i = 1$).

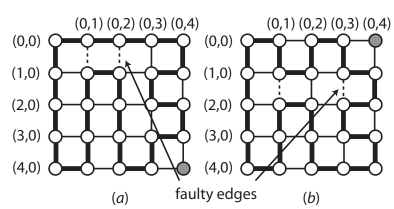

Figure 5. Cycles of length 24 in Q_2^5.

Suppose that at most 1 faulty edge is of the form $((i,j),(i+1,j))$, for any fixed $i \in \{0,1,2,3\}$. W.l.o.g., either there is a faulty edge e of the form $((0,j),(1,j))$ or there is a faulty edge e of the form $((1,j),(2,j))$ with the other fault f (different from $((0,0),(4,0))$) of the form $((2,j'),(3,j'))$. As above, we can construct healthy cycles of lengths 4, 6, 8 and 10 using the vertices of rows 0 and 1 or rows 1 and 2, respectively, so as to avoid e. W.l.o.g. the fault f different from e lies in column 0, 1 or 2. Depending upon where f lies, at least 1 of the cycles in Fig. 6(*a*), Fig. 6(*b*) and Fig. 6(*c*) can be progressively shortened so as to obtain healthy cycles of all lengths from 10 up to 24 and so that all faults are avoided (in these 3 figures: the different possibilities for the fault f are shown using dashed lines, with Fig. 6(*a*) and Fig. 6(*b*) depicting the situation when e joins vertices in rows 0 and 1 and Fig. 6(*c*) the situation when e joins vertices in rows 1 and 2; moreover, w.l.o.g. we have assumed that our cycle of length 10 spanning the vertices on rows 0 and 1 or on rows 1 and 2, respectively, omits edges joining column 4 vertices and column 0 vertices).

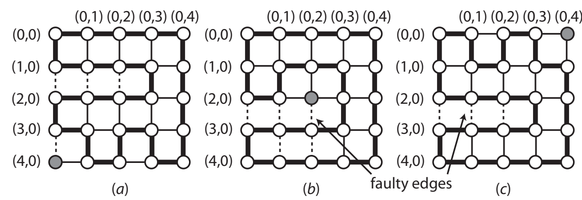

Figure 6. Cycles of length 24 in Q_2^5.

Case 2: exactly 2 faulty edges are column edges.

W.l.o.g. we may assume that the row fault is $((4,0),(4,4))$. If there is a column fault that is incident with a vertex in row 4 then w.l.o.g. we may assume that this fault joins a vertex in row 0 and a vertex in row 4 and that the other column fault lies in column 0, 1 or 2. This being the case, at least one of the cycles in Fig. 6(*a*) or Fig. 6(*b*) is healthy and yields healthy cycles of any even length l for which $4 \leq l \leq 24$ (note that if the column fault different from $((4,0),(4,4))$ is $((0,0),(1,0))$ then the cycle obtained by mapping the cycle in Fig. 6(*a*) according to the automorphism $(x,y) \mapsto (4-x,y)$ of Q_2^5 suffices).

Hence, we may assume that the row fault is $((4,0),(4,4))$ and that neither column fault is incident with a vertex on row 4. There exists an E-cycle rooted at $(0,m)$, for some $m \in \{0,1,2,3,4\}$, and spanning the vertices of $\{(i,j) : 0 \leq i \leq 3, 0 \leq j \leq 4\}$ that consists entirely of healthy edges. This E-cycle, of length 20, can clearly be extended using vertices in row 4 (no matter what the value of m) so as to obtain cycles of length 22 and 24, and the resulting cycle of length 24 can be progressively shortened so as to obtain healthy cycles of any even length l for which $4 \leq l \leq 18$. The result follows. $\square$

The proof of the following lemma is omitted as it can easily be obtained by hand by a case-by-case analysis (and the use of appropriate automorphisms).

Lemma 5 *Consider a* 3*-ary* 2*-cube* Q_2^3 *in which there are at most* 3 *faulty edges but where every vertex is incident with at least* 2 *healthy edges. There is a cycle of length* l *for every even* l *such that* $4 \leq l \leq 8$*.*

We draw the lemmas of this section together in the following corollary.

Corollary 6 *Let* $k \geq 3$*. Consider a* k*-ary* 2*-cube* Q_2^k *in which there are at most* 3 *faulty edges but where every vertex is incident with at least* 2 *healthy edges. There is a cycle of length* l *for every even* l *such that* $4 \leq l \leq k^2$*.*

When k is even then Corollary 6 is the best we can do, in the sense that as Q_2^k is bipartite, there cannot be any cycle of odd length. However, when k is odd we can say more.

Lemma 7 *Consider a* k*-ary* 2*-cube* Q_2^k*, where* $k \geq 7$ *is odd, in which there are at most* 3 *faulty edges but where every vertex is incident with at least* 2 *healthy edges. There is a cycle of length* l *for every odd* l *such that* $k \leq l \leq k^2$*.*

Proof Suppose that all faults are column faults. By proceeding as in the proof Case 1 of Lemma 3, there is a healthy cycle C, contained within the subgraph induced by the vertex set $\{(i,j) : 0 \leq i \leq k-2, 0 \leq j \leq k-1\}$, of every even length l for which $4 \leq l \leq k(k-1)$. Moreover, we may clearly arrange that every such cycle C contains an edge of the form $((k-2,j),(k-2,j+1))$ so that the path $((k-2,j),(k-1,j),(k-1,j+1),(k-2,j+1))$ is healthy. Thus, by joining any such cycle C (or any edge on row $k-2$) to a cycle of length k spanning the vertices of $\{(k-1,0),(k-1,1),\ldots,(k-1,k-1)\}$, we can clearly obtain a healthy cycle of every odd length l, where $k \leq l \leq k^2$.

Suppose that there is a row fault but that it is not the case that the faults form a path consisting of a column fault, followed by a row fault and ending with a column fault. Again, similarly to above, the proof of Case 2.1. of Lemma 3 suffices to enable us obtain a cycle of every odd length l, where $k \leq l \leq k^2$.

Finally, suppose that w.l.o.g. the faults are $((k-1,0),(0,0))$, $((0,0),(0,1))$ and $((0,1),(k-1,0))$ or the faults are $((k-1,0),(0,0))$, $((0,0),(0,1))$ and $((0,1),(1,1))$.

In the first case, we have a healthy E-cycle rooted at $(0,1)$ spanning the vertices of the first $k-1$ rows, which clearly suffices for us to construct a healthy cycle of every odd length l where $k \leq l \leq k^2$. In the second case, the healthy Hamiltonian cycle as depicted in Fig. 7 can be progressively shortened so as to obtain a healthy cycle of any odd length l where $3k-2 \leq l \leq k^2$ (whilst we have depicted the cycle only for Q_2^7, the construction in Q_2^k, where $k \geq 7$ is odd, should be clear). It is trivial to build healthy cycles of any odd length l for which $k \leq l \leq 3k-2$ and so the result follows. □

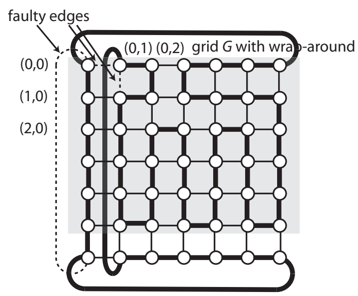

Figure 7. A Hamiltonian cycle in Q_2^7.

Lemma 8 *Consider a 5-ary 2-cube Q_2^5 in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges. There is a cycle of length l for every odd l such that $5 \leq l \leq 25$.*

Proof The proof is similar to that of Lemma 4 and so we do not present the full details. There exists some dimension containing at least 2 faulty edges, which w.l.o.g. we assume is dimension 2.

Case 1: all faulty edges are column edges.

W.l.o.g. we may assume that the edge $((0,0),(4,0))$ is faulty and that no other faulty edge joins a vertex in row 0 and a vertex in row $k-1$ (if there is a fault joining a vertex in row 0 and a vertex in row 4 then take the cycle in Fig. 8(a) and translate it, if necessary, to avoid any additional column fault before progressively shortening it).

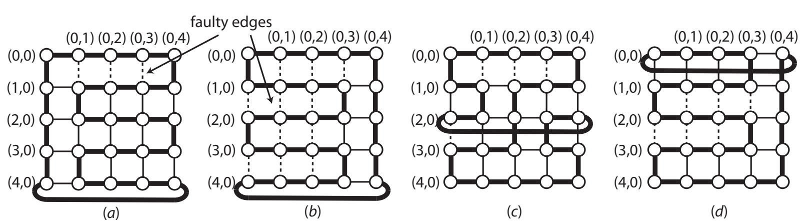

Figure 8. Cycles of length 25 in Q_2^5.

Suppose that 2 faulty edges are of the form $((i,j),(i+1,j))$, for some fixed $i \in \{0,1,2,3\}$. W.l.o.g. $i = 0$ or $i = 1$. We proceed as in the proof of Lemma 4 with the cycle as depicted in Fig. 8(*a*) (this cycle corresponds to the case when $i = 0$ and its image under the automorphism $(x, y) \mapsto (4 - x, y)$ to that when $i = 1$).

Suppose that at most 1 faulty edge is of the form $((i,j),(i+1,j))$, for any fixed $i \in \{0,1,2,3\}$. As in the proof of Lemma 4, w.l.o.g. either there is a faulty edge e of the form $((0,j),(1,j))$ or there is a faulty edge e of the form $((1,j),(2,j))$ with the other fault f (different from $((0,0),(4,0))$) of the form $((2,j'),(3,j'))$. Suppose that e is of the form $((0,j),(1,j))$. The cycles in Fig. 8(*b*) and Fig. 8(*c*) suffice (if $e = ((0,1),(1,1))$ and $f = ((2,1),(3,1))$ then take the image of the cycle in Fig. 8(*b*) under the automorphism $(x,y) \mapsto (x, 4-y)$). Suppose that e is of the form $((1,j),(2,j))$ with the other fault f of the form $((2,j'),(3,j'))$. The cycle in Fig. 8(*d*) suffices (if $e = ((1,3),(2,3))$ then progressively shorten to obtain a healthy cycle of any odd length from 25 down to 11, and then build healthy cycles of lengths 9, 7 and 5 separately).

<u>Case 2</u>: exactly 2 faulty edges are column edges.

W.l.o.g. we may assume that the row fault is $((0,0),(0,4))$. If there is a column fault e that is incident with a vertex in row 0 then w.l.o.g. we may assume that e joins a vertex in row 0 and a vertex in row 4 and that the other column f fault lies in column 0, 1 or 2. No matter where f lies, so long as it is not $((0,0),(1,0))$, one of the cycles in Fig. 8(*a*), Fig. 8(*b*) or Fig. 8(*c*) suffices. Suppose that $f = ((0,0),(1,0))$. The cycle in Fig. 9(*a*) suffices (note that $((0,0),(4,0))$ is necessarily healthy).

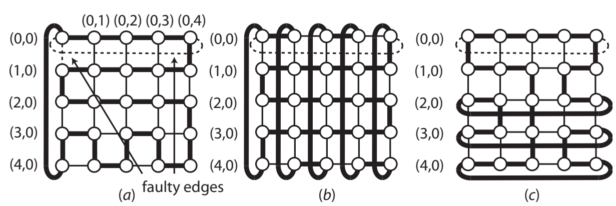

Figure 9. Cycles of length 25 in Q_2^5.

Hence, we may assume that the row fault is $((0,0),(0,4))$ and that neither column fault is incident with a vertex on row 0. Consider the cycle in Fig. 9(*b*) and its image under the automorphism $(x,y) \mapsto (x, 4-y)$. These cycles suffice to yield the result when the two column faults do not lie in columns 0 or 4 (as these cycles can be progressively shortened to cycles of length 5 no matter where the 2 column faults lie, subject to them both not lying in columns 0 or 4). If the 2 column faults lie in columns 0 or 4 then the cycle in Fig. 9(*c*) suffices to yield the result. The result follows. $\square$

We bring all the results of this section together in the following theorem.

Theorem 9 *Consider a k-ary 2-cube Q_2^k in which there are at most 3 faulty edges but where every vertex is incident with at least 2 healthy edges.*

(a) *If $k \geq 3$ then Q_2^k is bipancyclic.*

(b) *If $k \geq 5$ is odd then Q_2^k is k-pancyclic.*

An equivalent formulation of the above result is that Q_2^k is 3-edge-fault-tolerant bipancyclic, when $k \geq 3$, and 3-edge-fault-tolerant k-pancyclic, when $k \geq 5$ is odd, with both results assuming the conditional fault assumption that no vertex is incident with less than 2 healthy edges.

Theorem 9 cannot be improved when k is odd, for it is not difficult to see that when $n \geq 2$, Q_n^k has no odd length cycles of length less than k (see also [24]). Also, in Q_2^3 there are configurations of 3 faulty edges so that even though every vertex is incident with at least 2 healthy edges, no Hamiltonian cycle exists (one of these configurations is when the edges $((0,0),(0,1))$, $((0,1),(0,2))$, and $((0,2),(0,0))$ are faulty edges). We also note that (as was explained in [2]) Corollary 6 is optimal in the sense that there are configurations of 4 faults in Q_2^k for which a Hamiltonian circuit does not exist, no matter what the value of k (one such configuration is the set of faults $\{((0,0),(0,k-1)),((0,0),(k-1,0)),((1,1),(1,2)),((1,1),(2,1))\}$).

4 The general case

In this section, we prove our main result. The proof is long and complicated and so it might be beneficial if we outline our approach. Essentially, we proceed by induction and partition Q_n^k over a specific dimension so that we can ensure that there is a certain number of faults in this dimension (Theorem 9 deals with the base case of the induction). That leaves the rest of the faults spread over the k-ary $(n-1)$-cubes that result from the partition. We would like to apply the induction hypothesis to each of these k-ary $(n-1)$-cubes and then piece together the resulting cycles to achieve our required result. However, there are two cases to consider: the first is where, when we partition, there is some k-ary $(n-1)$-cube that does not satisfy our conditional fault assumption; and the second is where this is not the case. The second case is split into 2 further cases: when the faults not in the dimension over which we have partitioned are not co-located in the same k-ary $(n-1)$-cube; and the second case is when they are. Throughout, we build different healthy cycles of different (even) lengths, in a very non-uniform fashion and using a variety of techniques.

Theorem 10 *Let $n \geq 2$ and let $k \geq 4$ be even. Suppose that the k-ary n-cube Q_n^k has at most $4n-5$ faulty edges but is such that every vertex is incident with at least 2 healthy edges. Then Q_n^k contains cycles of any even length from 4 up to k^n; that is, Q_n^k is $(4n-5)$-edge-fault-tolerant-bipancyclic under the conditional fault assumption that every vertex is incident with at least 2 healthy edges.*

Proof Let $n \geq 3$ throughout and suppose as our induction hypothesis that the result holds for Q_{n-1}^k. The base case of our induction follows from Theorem 9. Suppose that Q_n^k has $4n-5$ faulty edges so that every vertex is incident with at least 2 healthy edges. There exists some dimension $i \in \{1, 2, \ldots, n\}$ such that dimension i contains at least 3 faults; w.l.o.g. we may assume that dimension 1 contains at least 3 faults. Partition Q_n^k over dimension 1 to obtain $Q_0, Q_1, \ldots, Q_{k-1}$. There are at most $4n-8$ faults not contained in dimension 1. In each of the cases below, we construct healthy cycles of various lengths in a piecemeal fashion and using a number of different constructions.

Case 1: there exists some vertex x in some Q_i, where $i \in \{0, 1, \ldots, k-1\}$, such that x is incident with at least $2n-3$ faults in Q_i.

W.l.o.g. we may assume that x lies in Q_0. Note that x is the only vertex that is incident with at least $2n-3$ faults in the Q_i in which it lies (as otherwise we would have $4n-7$ faults not lying in dimension 1). Suppose that for every pair of neighbours y and z of x in Q_0, with $y \neq z$, at least 1 of the edges $(y, n_1(y))$ and $(z, n_1(z))$ is faulty and at least 1 of the edges $(y, n_{k-1}(y))$ and $(z, n_{k-1}(z))$ is faulty. So, there must be at least $(2n-3)+(2n-3)+(2n-3) = 6n-9 > 4n-5$ faults in total, which yields a contradiction. W.l.o.g. we may assume that there are distinct edges (x, y) and (x, z) in Q_0 such that $(y, n_1(y))$ and $(z, n_1(z))$ are healthy. Amend Q_0 as follows so as to obtain $\tilde{Q}_0$.

- If x is incident with a healthy edge (x, w) in Q_0 and $y \neq w \neq z$ then make (x, w) faulty and make (x, y) and (x, z) healthy.
- If (x, y) (resp. (x, z)) is healthy then make (x, z) (resp. (x, y)) healthy.
- If x is incident only with faults in Q_0 then make (x, y) and (x, z) healthy.

Note that in $\tilde{Q}_0$, vertex x is incident with 2 healthy edges and there are at most $4n-9$ faults. Suppose that some other vertex u of Q_0 is incident with at most 1 healthy edge in $\tilde{Q}_0$. So, we must have that (x, u) is an edge that is healthy in Q_0 but which is made faulty in $\tilde{Q}_0$. Thus in Q_0, (x, u) is an edge that is the only healthy edge incident with x and 1 of 2 healthy edges incident with u, with the result that Q_0 has at least $4n-7$ faults, which yields a contradiction. Hence, we can apply the induction hypothesis to $\tilde{Q}_0$ and so obtain a Hamiltonian cycle C_0 in Q_0 containing the (potentially faulty) edges (x, y) and (x, z) and where all other edges of C_0 are healthy (in Q_0).

Consider Q_1, which contains at most $2n - 5$ faults. The vertex $n_1(x)$ is incident with at most $2n - 2$ healthy edges. We obtain $\tilde{Q}_1$ by making all healthy edges incident with $n_1(x)$ faulty, apart from $(n_1(x), n_1(y))$ and $(n_1(x), n_1(z))$ which we make healthy if necessary. This means introducing at most $2n - 4$ faults, and so $\tilde{Q}_1$ has at most $4n - 9$ faults. Suppose that $\tilde{Q}_1$ has a vertex that is incident with at most 1 healthy edge in $\tilde{Q}_1$. As any vertex of Q_1 is incident with at least 2 healthy edges in Q_1, this means that Q_1 has at least $2n - 4$ faults, which yields a contradiction. Thus, we can apply the induction hypothesis to $\tilde{Q}_1$ to obtain a Hamiltonian cycle C_1 in Q_1 that contains the edges $(n_1(x), n_1(y))$ and $(n_1(x), n_1(z))$ and where all other edges of C_1 are healthy (in Q_1). We can join C_0 and C_1 by removing the edges (x, y), (x, z), $(n_1(x), n_1(y))$ and $(n_1(x), n_1(z))$ and including the edges $(y, n_1(y))$ and $(z, n_1(z))$ to obtain a cycle C_{01}, spanning all vertices of Q_0 and Q_1 apart from x and $n_1(x)$, that has length $2k^{n-1} - 2$ and which only contains healthy edges.

In the rest of this case, we construct cycles of every even length m, where $4 \leq m \leq k^n$, with the cycle C_{01} providing a base cycle from which to work in many situations. Moreover, we do this for batches of values for m. For example, our first batch of values, below, is $3k^{n-1} - 2 \leq m \leq (k-1)k^{n-1}$ and our second is $(k-1)k^{n-1} \leq m \leq k^n - (4n-2)$; eventually, we cover $4 \leq m \leq k^n$ (throughout, m is always even). To aid readability, we partition our constructions according to the techniques used. We remind the reader that $k \geq 4$ is even and $n \geq 3$.

<u>Case 1.1</u>: Consider the path P_1 of length $k^{n-1} - 2$ from $n_1(y)$ to $n_1(z)$ on C_1. By partitioning the vertices on this path into batches of 3 consecutive vertices and noting that $\lfloor \frac{k^{n-1}-1}{3} \rfloor > 2n - 2$, where $2n - 2$ is an upper bound on the number of faults not in Q_0, there are edges (u, v) and (v, w) of P_1 such that all edges of

$$\begin{aligned}\{&(n_i(u), n_{i+1}(u)), (n_i(v), n_{i+1}(v)), (n_i(w), n_{i+1}(w)), (n_i(u), n_i(v)),\\ &(n_i(v), n_i(w)) : 1 \leq i \leq k-1\}\end{aligned}$$

are healthy.

Fix $\alpha \in \{2, 3, \ldots, k-1\}$ and let $i \in \{2, 3, \ldots, \alpha\}$. In Q_i make all edges incident with $n_i(v)$ faulty apart from the edges $(n_i(u), n_i(v))$ and $(n_i(v), n_i(w))$, which are healthy, and denote the amended Q_i by $\tilde{Q}_i$. Note that $\tilde{Q}_i$ has at most $(2n-5) + (2n-4) = 4n - 9$ faults. Also, if $\tilde{Q}_i$ has a vertex that is incident with at most 1 healthy edge in $\tilde{Q}_i$ then this means that Q_i has at least $2n - 4$ faults, which yields a contradiction. By the induction hypothesis applied to $\tilde{Q}_i$, we obtain a healthy

Hamiltonian cycle C_i in Q_i that contains $(n_i(u), n_i(v))$ and $(n_i(v), n_i(w))$. We can join the cycles $C_{01}, C_2, C_3, \ldots, C_\alpha$ using healthy dimension 1 edges, as appropriate, to obtain a cycle D_α of length $(\alpha+1)k^{n-1}-2$ spanning all vertices of $Q_0, Q_1, \ldots, Q_\alpha$ apart from x and $n_1(x)$. The situation can be depicted as in Fig. 10 (where α happens to be odd).

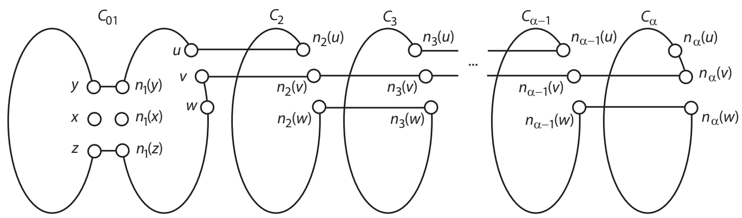

Figure 10. Joining cycles together when α is odd.

Suppose that $\alpha \in \{2, 3, \ldots, k-3\}$ and that $\beta \in \{0, 1, \ldots, \frac{k^{n-1}}{2}+1\}$. Let P_0 (resp. P_α) be the path of length $k^{n-1}-2$ (resp. $k^{n-1}-1$) on C_{01} (resp. C_α) from y to z (resp. from $n_\alpha(v)$ to $n_\alpha(w)$, if α is odd, and from $n_\alpha(v)$ to $n_\alpha(u)$, if α is even). By considering alternating edges of P_0 and P_α, there are at least $\frac{k^{n-1}-2}{2} + \frac{k^{n-1}}{2} = k^{n-1}-1$ mutually non-incident edges of P_0 and P_α. Count the number of such edges (s,t) for which the path $(s, n_{k-1}(s), n_{k-1}(t), t)$ is healthy, if (s,t) lies on P_0, or for which the path $(s, n_{\alpha+1}(s), n_{\alpha+1}(t), t)$ is healthy, if (s,t) lies on P_α. This number is at least $k^{n-1}-1-(2n-2) > \beta$ and so we can choose β such edges (s,t) and easily extend D_α, using the appropriate healthy paths of length 3, to obtain a healthy cycle of length $(\alpha+1)k^{n-1}-2+2\beta$ (note that $k-1 \neq \alpha+1$); that is, we have constructed healthy cycles of any even length from $3k^{n-1}-2$ up to $(k-1)k^{n-1}$.

Suppose that $\alpha = k-2$ and $\beta \in \{0, 1, \ldots, \frac{k^{n-1}}{2} - (2n-2)\}$; thus, D_α has length $(k-1)k^{n-1}-2$. By considering alternate edges on P_{k-2} (as defined in the previous paragraph), there are $\frac{k^{n-1}}{2}$ mutually non-incident edges of P_{k-2}. Count the number of such edges (s,t) for which the path $(s, n_{k-1}(s), n_{k-1}(t), t)$ is healthy. This number is at least $\frac{k^{n-1}}{2} - (2n-2) \geq \beta$ and so we can choose β such edges (s,t) and easily extend D_α, using the appropriate healthy paths of length 3, to obtain a healthy cycle of length $(k-1)k^{n-1}-2+2\beta$; that is, we have constructed healthy cycles of any even length from $(k-1)k^{n-1}-2$ up to $k^n-(4n-2)$.

<u>Case 1.2</u>: We shall now construct healthy cycles of any even length from 4 up to $2k^{n-1}$. By the induction hypothesis applied to Q_1, there is a healthy cycle of any even length from 4 up to k^{n-1}. Let C_1' be a healthy Hamiltonian cycle in Q_1. By considering alternating edges on C_1', we have $\frac{k^{n-1}}{2}$ mutually non-incident edges on C_1'. For each such edge (s,t), let the set of edges $T_{s,t} = \{(n_i(s), n_{i+1}(s)), (n_i(t),$

$n_{i+1}(t)), (n_{i+1}(s), n_{i+1}(t)) : i = 1, 2, \ldots, k-2\}$. Note that if all edges of some $T_{s,t}$ are healthy then we can extend C'_1 to obtain healthy cycles of lengths $k^{n-1}+2, k^{n-1}+4, \ldots, k^{n-1}+2(k-2)$ by replacing the edge (s,t) with the paths $(s, n_2(s), n_2(t), t)$, $(s, n_2(s), n_3(s), n_3(t), n_2(t), t)$ and so on. At least $\frac{k^{n-1}}{2} - (2n-2)$ of these $T_{s,t}$'s are such that all of the edges in $T_{s,t}$ are healthy, and so we can obtain healthy cycles of any even length from k^{n-1} up to $k^{n-1} + 2(k-2)(\frac{k^{n-1}}{2} - (2n-2)) = (k-1)k^{n-1} - 2(k-2)(2n-2) > k^{n-1} + (4n-4)$. Alternatively, suppose that $2n-1 \leq \beta \leq \frac{k^{n-1}}{2}$. By the induction hypothesis applied to Q_1, there is a cycle C''_1 of length 2β in Q_1. As $\beta > 2n-2$, there is an edge (s,t) of C''_1 such that both $(s, n_2(s))$ and $(t, n_2(t))$ are healthy. Amend Q_2 so as to ensure that exactly 2 edges incident with $n_2(s)$ are healthy, one of which is $(n_2(s), n_2(t))$ and the other of which is a healthy edge of Q_2, and denote this amended version of Q_2 by $\tilde{Q}_2$. It is also the case that every vertex of $\tilde{Q}_2$ is incident with at least 2 healthy edges in $\tilde{Q}_2$. By the induction hypothesis applied to $\tilde{Q}_2$, there is a Hamiltonian path P_2 in Q_2 from $n_2(s)$ to $n_2(t)$ consisting of healthy edges. Thus, we have a healthy cycle of length $k^{n-1}+2\beta$ formed by joining C''_1 to P_2. That is, we have constructed a healthy cycle of every even length from $k^{n-1} + (4n-2)$ up to $2k^{n-1}$. Hence, we have constructed a healthy cycle of every even length from 4 up to $2k^{n-1}$.

We can extend any 1 of the cycles constructed in the previous paragraph as follows. In the first construction, instead of starting with the cycle C'_1, start with the cycle C_{01}, of length $2k^{n-1}-2$, as constructed earlier. By applying identical reasoning (but noting that at least $\frac{k^{n-1}}{2} - 1 - (2n-2)$ of the $T_{s,t}$'s are such that all of the edges in $T_{s,t}$ are healthy), we obtain a healthy cycle of any even length from $2k^{n-1}$ up to $2k^{n-1} - 2 + 2(k-2)(\frac{k^{n-1}}{2} - 1 - (2n-2)) \geq 2k^{n-1} - 2 + 4(\frac{4^{n-1}}{2} - 1 - (2n-2)) = 2k^{n-1} + 2.4^{n-1} + 2 - 8n \geq 2k^{n-1} + (4n-4)$. Alternatively, suppose that $2n-1 \leq \beta \leq \frac{k^{n-1}}{2}$. Instead of starting with the cycle C''_1, of length 2β, as in the previous paragraph, start with a cycle, as constructed in the previous paragraph, of length $k^{n-1}+2\beta$ (recall, this cycle was obtained by joining C''_1 to P_2) and extend this cycle just as we did above but using a Hamiltonian path in Q_3. That is, we have constructed a healthy cycle of every even length from $2k^{n-1} + (4n-2)$ up to $3k^{n-1}$. Hence, taking into account our earlier constructions (above and in Case 1.1), we have constructed a healthy cycle of every even length from 4 up to $k^n - (4n-2)$.

Case 1.3: Thus, all that remains is for us to build healthy cycles of any even length from $k^n - (4n-4)$ up to k^n (of course, by [2] there is a healthy Hamiltonian cycle in Q^k_n). Let $\beta \in \{\frac{k^{n-1}-(4n-4)}{2}, \frac{k^{n-1}-(4n-6)}{2}, \ldots, \frac{k^{n-1}}{2}\}$. By arguing exactly as we did earlier, we can apply the induction hypothesis to $\tilde{Q}_0$ (as constructed earlier) and obtain a cycle C'_0

in Q_0 of length 2β (note that $2\beta > 4$). There are two possibilities: either the cycle C'_0 contains both (x, y) and (x, z); or the cycle C'_0 does not contain x. In the former case, we proceed exactly as we did earlier. First, we build a cycle C'_{01} of length $k^{n-1}+2\beta-2$ spanning all vertices of Q_1 apart from $n_1(x)$ and all vertices of C'_0 apart from x; and, second, we extend this cycle C'_{01} (as we did in Fig. 10) using (healthy) Hamiltonian cycles in each of $Q_2, Q_3, \ldots, Q_{k-1}$. Thus, we obtain a healthy cycle in Q^k_n of length $(k-2)k^{n-1}+k^{n-1}+2\beta-2 = (k-1)k^{n-1}+2\beta-2$; that is, we have constructed healthy cycles of any even length from $k^n - (4n - 2)$ up to $k^n - 2$. Consequently, we may assume that the cycle C'_0 does not pass through the vertex x. By considering alternating edges on C'_0, there is a set X of β mutually non-incident edges (s, t) of C'_0. Consider the set of 2β paths $\{(s, n_1(s), n_1(t), t), (s, n_{k-1}(s), n_{k-1}(t), t) : (s, t) \in X\}$. The number of faults not in Q_0 is at most $2n-2$ and as $2\beta > 2n-2$, w.l.o.g. we may assume that there is an edge (s, t) of C'_0 such that the edges of the path $(s, n_1(s), n_1(t), t)$ are healthy. Just as we have done throughout this proof, we can build a healthy Hamiltonian cycle C'_1 in Q_1 containing the edge $(n_1(s), n_1(t))$ and join C'_0 and C'_1 to obtain a healthy cycle C'_{01} of length $k^{n-1} + 2\beta$. By continuing this argument iteratively, we obtain a healthy cycle of length $(k - 1)k^{n-1} + 2\beta$; that is, irrespective of whether the vertex x lies on C'_0, we have healthy cycles of any even length from $k^n - (4n - 4)$ up to k^n.

Case 2: every vertex x in any Q_i, where $i \in \{0, 1, \ldots, k-1\}$, is such that x is incident with at least 2 healthy edges in Q_i.

As in Case 1, we construct cycles of various lengths in batches. There are two possibilities: either every Q_i, where $i \in \{0, 1, \ldots, k - 1\}$, contains at most $4n - 9$ faults; or w.l.o.g. Q_0 contains $4n - 8$ faults.

Case 2.1: every Q_i, where $i \in \{0, 1, \ldots, k - 1\}$, contains at most $4n - 9$ faults.

W.l.o.g. suppose that Q_0 contains most faults from $Q_0, Q_1, \ldots, Q_{k-1}$. In particular, if $i \in \{1, 2, \ldots, k - 1\}$ then Q_i contains at most $2n - 4$ faults.

Case 2.1.1: no Q_i, where $i \in \{1, 2, \ldots, k - 1\}$, contains more than $2n - 5$ faults.

By the induction hypothesis applied to Q_0, there are healthy cycles of any even length from 4 up to k^{n-1}. In particular, there is a healthy Hamiltonian cycle C_0 in Q_0. By considering alternating edges on C_0, we have $\frac{k^{n-1}}{2}$ mutually non-incident edges on C_0. For each such edge (s, t), let the set of edges $T_{s,t} = \{(n_i(s), n_{i+1}(s)), (n_i(t), n_{i+1}(t)), (n_{i+1}(s), n_{i+1}(t)) : i = 0, 1, \ldots, k - 1\}$. Note that if all edges of some $T_{s,t}$ are healthy then we can extend C_0 to obtain healthy cycles of lengths $k^{n-1}+2, k^{n-1}+4, \ldots, k^{n-1}+2(k-1)$ by replacing the edge (s, t) with the paths $(s, n_1(s), n_1(t), t), (s,$

$n_1(s), n_2(s), n_2(t), n_1(t), t)$ and so on. Alternatively, we can obtain our healthy cycles by extending C_0 by replacing the edge (s,t) with the paths $(s, n_{k-1}(s), n_{k-1}(t), t), (s, n_{k-1}(s), n_{k-2}(s), n_{k-2}(t), n_{k-1}(t), t)$ and so on. In fact, if there is only 1 faulty edge in $T_{s,t}$ then we can clearly still extend C_0 using healthy paths of lengths $3, 5, \ldots, 2(\frac{k}{2}-1)+1 = k-1$. So, let α be the number of $T_{s,t}$'s that contain exactly 1 fault and let β be the number of $T_{s,t}$'s that contain at least 2 faults. By extending C_0 using different paths, we can clearly obtain healthy cycles of all even lengths from 4 up to $k^{n-1} + 2(k-1)(\frac{k^{n-1}}{2} - (\alpha+\beta)) + \alpha(k-2) = k^n - 2(k-1)\beta - k\alpha$. As $\alpha + 2\beta \leq 4n-5$, we have that $\beta \leq \frac{(4n-5-\alpha)}{2}$, and so $k^n - 2(k-1)\beta - k\alpha \geq k^n - 2(k-1)\frac{(4n-5-\alpha)}{2} - k\alpha = k^n - \alpha - (k-1)(4n-5) \geq k^n - k(4n-5) \geq 2k^{n-1}$. Hence, we have constructed a healthy cycle of every even length from 4 up to $2k^{n-1}$.

Let $\alpha \in \{1, 2, \ldots, k-2\}$. By the induction hypothesis, Q_0 has a healthy Hamiltonian cycle C_0. By considering alternating edges on C_0, there are $\frac{k^{n-1}}{2}$ mutually non-incident edges of C_0. Count the number of such edges (s,t) for which at least 1 of $(s, n_1(s))$ and $(t, n_1(t))$ is faulty. This cannot be more than $4n-5$. However, as $\frac{k^{n-1}}{2} > 4n-5$, there is at least one edge (s,t) of C_0 for which $(s, n_1(s))$ and $(t, n_1(t))$ are both healthy. Consider the vertex $n_1(s)$ in Q_1. Amend Q_1 by ensuring that $2n-4$ edges incident with $n_1(s)$ are faulty so that $(n_1(s), n_1(t))$ and exactly 1 other edge incident with $n_1(s)$ are healthy, and denote the amended version of Q_1 by $\tilde{Q}_1$. Thus, $\tilde{Q}_1$ has at most $4n-9$ faults. Note that as Q_1 contains at most $2n-5$ faults then every vertex in $\tilde{Q}_1$ is incident with at least 2 healthy edges. By the induction hypothesis applied to $\tilde{Q}_1$, there is a healthy Hamiltonian path in Q_1 from $n_1(s)$ to $n_1(t)$. Thus, we have a healthy cycle of length $2k^{n-1}$ spanning all vertices in Q_0 and Q_1. We can continue iteratively in this way (and as we have done previously) so as to obtain a healthy cycle D_α of length $(\alpha+1)k^{n-1}$ spanning the vertices of $Q_0, Q_1, \ldots, Q_\alpha$.

Suppose that $\alpha \neq k-2$ and let $\beta \in \{0, 1, \ldots, \frac{k^{n-1}}{2}\}$. Let P_0 (resp. P_α) be the sub-path of D_α spanning the vertices of Q_0 (resp. Q_α). Both of these paths have length $k^{n-1}-1$. By considering alternating edges on P_0 and P_α, there are k^{n-1} mutually non-incident edges of P_0 and P_α. As $k^{n-1} - (4n-5) > \frac{k^{n-1}}{2}$, we can choose β mutually non-incident such edges (s,t) so that either the path $(s, n_{\alpha+1}(s), n_{\alpha+1}(t), t)$ or the path $(s, n_{k-1}(s), n_{k-1}(t), t)$ is healthy, depending upon whether (s,t) lies on P_α or P_0, respectively (note that $k-1 \neq \alpha+1$). Consequently, we can clearly obtain a healthy cycle in Q_n^k of length $(\alpha+1)k^{n-1} + 2\beta$; that is, we have cycles of any even length from $2k^{n-1}$ up to $(k-1)k^{n-1}$.

Suppose that $\alpha = k-2$ and let $\beta \in \{0, 1, \ldots, \frac{k^{n-1}}{2} - (4n-5)\}$. There are $\frac{k^{n-1}}{2}$ mutually non-incident edges on the sub-path P_{k-2} of D_{k-2} spanning the vertices of Q_{k-2}. Just as in the previous paragraph, we can choose β mutually non-incident edges (s,t) on P_{k-2} so that the path $(s, n_{k-1}(s), n_{k-1}(t), t)$ is healthy. Thus, we have healthy cycles of any even length from $(k-1)k^{n-1}$ up to $k^n - (8n-10)$. In fact, if the number of faults joining a vertex in Q_{k-2} to a vertex in Q_{k-1} plus the number of faults in Q_{k-1} is γ then we have healthy cycles of any even length from 4 up to $k^n - 2\gamma$. We shall return to this comment in a moment.

All that remains is for us to obtain healthy cycles of any even length from $k^n - (8n-12)$ up to k^n. Suppose that $\frac{k^{n-1}}{2} - (4n-5) + 3 \leq \beta \leq \frac{k^{n-1}}{2}$. By the induction hypothesis applied to Q_0, there is a healthy cycle C_0 of length 2β in Q_0. By considering alternating edges on C_0, there is a set X of β mutually non-incident edges of C_0. For any edge $(s,t) \in X$, define the path $\rho_{k-1}(s,t) = (s, n_{k-1}(s), n_{k-1}(t), t)$ and the path $\rho_1(s,t) = (s, n_1(s), n_1(t), t)$, and count the number of such paths that contain at least 1 fault. This number is at most $4n-5$. So, if $2\beta > 4n-5$ then we can find a path $\rho_{k-1}(s,t)$ or $\rho_1(s,t)$ that consists entirely of healthy edges. However, $2\beta \geq k^{n-1} - (8n-10) + 6 > 4n-5$, and so w.l.o.g. there is an edge (s,t) of C_0 so that the path $\rho_1(s,t) = (s, n_1(s), n_1(t), t)$ consists entirely of healthy edges. We can amend Q_1 to obtain $\tilde{Q}_1$ so that $n_1(s)$ is incident with exactly $2n-4$ faults in $\tilde{Q}_1$, one of which is $(n_1(s), n_1(t))$. Thus, $\tilde{Q}_1$ has at most $4n-9$ faults. Moreover, every vertex in $\tilde{Q}_1$ is incident with at least 2 healthy edges. By the induction hypothesis applied to $\tilde{Q}_1$, there is a healthy Hamiltonian path from $n_1(s)$ to $n_1(t)$ in Q_1. We can join this path with C_0, using the healthy edges $(s, n_1(s))$ and $(t, n_1(t))$, so as to obtain a healthy cycle C_{01} of length $k^{n-1} + 2\beta$. As $\frac{k^{n-1}}{2} > 4n-5$, we can iteratively extend C_{01} to a cycle of length $(k-1)k^{n-1} + 2\beta$; that is, we have healthy cycles of any even length from $k^n - (8n-16)$ up to k^n.

Thus, we only have to find healthy cycles of lengths $k^n - (8n-12)$ and $k^n - (8n-14)$. From our comment above, relating to the number γ of faults joining vertices in Q_{k-2} and Q_{k-1} or lying in Q_{k-1}, we may assume that γ is $4n-5$ or $4n-6$. By the induction hypothesis applied to Q_0, we can find healthy cycles C_0' and C_0'' of lengths $k^{n-1} - 8n + 12$ and $k^{n-1} - 8n + 14$, respectively. As all but at most 1 fault is incident with a vertex of Q_{k-1}, there clearly exists an edge (s,t) of C_0' or C_0'' such that $(s, n_1(s))$ and $(t, n_1(t))$ are both healthy. Just as we have done a number of times so far, we can iteratively extend C_0' and C_0'' by using appropriately chosen Hamiltonian cycles in $Q_1, Q_2, \ldots, Q_{k-1}$ so as to build healthy cycles in Q_n^k of lengths $k^n - (8n-12)$ and $k^n - (8n-14)$. Thus, we have constructed healthy cycles of any even length from 4 up to k^n.

<u>Case 2.1.2</u>: some Q_i, where $i \in \{1, 2, \ldots, k-1\}$, contains $2n-4$ faults.

It must be the case that Q_0 contains $2n-4$ faults, Q_i contains $2n-4$ faults and this accounts for all faults in Q_n^k apart from the 3 faults in dimension 1. W.l.o.g. we may assume that Q_1 contains no faults. By the induction hypothesis applied to Q_0, there is a healthy cycle of length 2β in Q_0, for any $\beta \in \{2, 3, \ldots, \frac{k^{n-1}}{2}\}$. Let C_0 be the cycle of length k^{n-1} in Q_0 that we have just constructed and let x, y and z be consecutive vertices on this cycle so that all edges of $\{(n_j(x), n_{j+1}(x)), (n_j(y), n_{j+1}(y)), (n_j(z), n_{j+1}(z)), (n_j(x), n_j(y)), (n_j(y), n_j(z)) : j = 0, 1, \ldots, k-1\}$ are healthy. Such consecutive vertices exist when $k > 4$ or $n > 3$ as $\lfloor \frac{k^{n-1}}{3} \rfloor > (2n-4) + 3$. Suppose that $k = 4$ and $n = 3$ and that there do not exist consecutive vertices x, y and z with the properties as stated. Note that there are 5 faults not lying in Q_0. Enumerate the vertices of C_0 as $u_0, u_1, \ldots, u_{15}$, and for $0 \leq l \leq 15$, let T_l be the set of edges $\{(n_j(u_l), n_{j+1}(u_l)), (n_j(u_{l+1}), n_{j+1}(u_{l+1})), (n_j(u_{l+2}), n_{j+1}(u_{l+2})), (n_j(u_l), n_j(u_{l+1})), (n_j(u_{l+1}), n_j(u_{l+2})) : j = 0, 1, \ldots, k-1\}$ (with addition on the indices of the u_l's modulo 15). So, each of T_0, T_3, T_6, T_9 and T_{12} (which are mutually disjoint as sets of edges) must contain a fault, and this accounts for all 5 faults. Also, T_{14} must contain a fault and so w.l.o.g. T_{12} must contain a dimension 1 fault of the form $(n_j(u_{14}), n_{j+1}(u_{14}))$. As T_{11} must contain a fault, T_9 must contain a dimension 1 fault of the form $(n_j(u_{11}), n_{j+1}(u_{11}))$. Arguing in this way yields that there must be more than 3 dimension 1 faults which yields a contradiction. Hence, we can find x, y and z as required.

Amend Q_i so that $n_i(y)$ is incident with exactly 2 healthy edges, namely the edges $(n_i(x), n_i(y))$ and $(n_i(y), n_i(z))$ which are healthy in Q_i. Denote this amended version of Q_i by $\tilde{Q}_i$. Note that $\tilde{Q}_i$ has at most $4n-8$ faults. Suppose that $\tilde{Q}_i$ has at most $4n-9$ faults and there is a vertex that is incident with at most 1 healthy edge. This vertex must be a neighbour of $n_i(y)$ so that this edge is healthy in Q_i and, further, it must be incident with $2n-4$ faults in Q_i. So, in order to form $\tilde{Q}_i$ we must have introduced $2n-4$ faults which yields a contradiction as $\tilde{Q}_i$ only has $4n-9$ faults. Thus, if $\tilde{Q}_i$ has at most $4n-9$ faults then every vertex of $\tilde{Q}_i$ is incident with at least 2 healthy edges. Alternatively, suppose that $\tilde{Q}_i$ has $4n-8$ faults; so, we have made $2n-4$ edges incident with $n_i(y)$ faulty (all except $(n_i(x), n_i(y))$ and $(n_i(y), n_i(z))$). In this case, there might be a vertex w of $\tilde{Q}_i$, adjacent to $n_i(y)$ and different from $n_i(x)$ and $n_i(z)$, such that w is incident with exactly 1 healthy edge in $\tilde{Q}_i$. If such a vertex w exists then let the edge $(w, n_i(y))$ revert back to being healthy in $\tilde{Q}_i$; otherwise, choose any faulty edge $(w, n_i(y))$, where $n_i(x) \neq w \neq n_i(z)$, and let it revert back to being healthy in $\tilde{Q}_i$. Denote $\tilde{Q}_i$ after any additional amendments by $\hat{Q}_i$ (note that $\hat{Q}_i$ contains at most $4n-9$ faults).

We can now apply the induction hypothesis to $\tilde{Q}_i$ or $\hat{Q}_i$, as appropriate, so as to obtain a cycle C_i of length k^{n-1}. If we are working with $\tilde{Q}_i$ then C_i contains the edges $(n_i(x), n_i(y))$ and $(n_i(y), n_i(z))$ and all edges of C_i are healthy in Q_i; if we are working with $\hat{Q}$ then C_i contains at least 1 of $(n_i(x), n_i(y))$ and $(n_i(y), n_i(z))$ and all edges of C_i are healthy in Q_i. In the latter case, w.l.o.g. we may assume that C_i contains $(n_i(x), n_i(y))$. Hence, whatever the case, we may assume that the cycle C_i contains the edge $((n_i(x), n_i(y))$ and that every edge of this cycle is healthy in Q_i.

For each $j \in \{0, 1, \ldots, k-1\} \setminus \{0, i\}$, let C_j be the isomorphic copy of C_0 in Q_j. For any $l \in \{1, 2, \ldots, k\}$, we can clearly join the cycles $C_i, C_{i+1}, \ldots, C_{i+l-1}$ (with arithmetic on indices modulo k) similarly to as is depicted in Fig. 10 so as to obtain a cycle D_l of length $l.k^{n-1}$, spanning all vertices of $Q_i, Q_{i+1}, \ldots, Q_{i+l-1}$, that is healthy in Q_n^k. Fix any $l \in \{1, 2, \ldots, k-1\}$ and choose some edge (s, t) of D_l that lies in Q_i and for which $(n_i(s), n_{i-1}(s))$ and $(n_i(t), n_{i-1}(t))$ are healthy edges (such an edge clearly exists as $\frac{k^{n-1}}{2} > 3$). By [24], Q_{i-1} is edge-bipancyclic (note that there are no faults in Q_{i-1}) and so contains a (healthy) cycle of length 2β, for any $\beta \in \{2, 3, \ldots, \frac{k^{n-1}}{2}\}$, that contains the edge $(n_{i-1}(s), n_{i-1}(t))$. Thus, we obtain a healthy cycle in Q_n^k of every even length m where $4 \leq m \leq k^n$.

Case 2.2: Q_0 contains $4n-8$ faults.

Choose some fault (x, y) in Q_0 such that the path $(s, n_1(s), n_1(t), t)$ is healthy and amend Q_0 so as to make this fault healthy. Applying the induction hypothesis to this amended version of Q_0 yields a healthy Hamiltonian path P_0 in Q_0 from x to y. We can join this healthy Hamiltonian path in Q_0 to its isomorphic copy in Q_1, namely a healthy Hamiltonian path P_1 from $n_1(x)$ to $n_1(y)$. Thus, we can join P_0 and P_1 to obtain a healthy cycle of length $2k^{n-1}$. Let (s, t) be some edge of P_1. We can replace (s, t) with the paths $(s, n_2(s), n_2(t), t)$, $(s, n_2(s), n_3(s), n_3(t), n_2(t), t)$, and so on, so as to obtain healthy cycles of lengths $2k^{n-1}+2, 2k^{n-1}+4, \ldots, 2k^{n-1}+2(k-2)$. Choosing other such edges and extending in the same way clearly enables us to build healthy cycles of any length from $2k^{n-1}$ up to k^n.

Consider Q_1 and Q_2. Neither contains a fault and there are at most 3 faults joining a vertex in Q_1 to a vertex in Q_2. Applying the induction hypothesis to Q_1 yields healthy cycles of all even lengths from 4 up to k^{n-1}. Let $\beta \in \{4, 5, \ldots, \frac{k^{n-1}}{2}\}$ and let C_1 be the cycle in Q_1 of length 2β just constructed. There is an edge (s, t) of C' such that the edges $(s, n_2(s))$ and $(t, n_2(t))$ are both healthy. By applying the induction hypothesis to an appropriately amended version of Q_2, we can obtain a healthy Hamiltonian cycle C_2 in Q_2 containing the edge $(n_2(s), n_2(t))$. Hence, we obtain healthy cycles of all even lengths from $k^{n-1}+8$ up to $2k^{n-1}$. In order to obtain cycles of lengths $k^{n-1}+2$, $k^{n-1}+4$ and $k^{n-1}+6$, we extend a healthy Hamiltonian cycle in Q_1 by replacing up to 3 edges of this cycle of the form (s, t) with healthy paths of the form

$(s, n_2(s), n_2(t), t)$. The result follows. $\square$

It was proven in [2] that there are configurations of $4n-4$ faulty edges in Q_n^k, where $k \geq 3$, so that even if every vertex is incident with at least 2 healthy edges, there does not exist a Hamiltonian cycle. Such a configuration is obtained by taking a cycle (x, y, u, v) in Q_n^k and ensuring that all edges incident with x are faulty apart from (x, y) and (x, u) and that all edges incident with u are faulty apart from (u, y) and (u, v). This amounts to $4n-4$ faults. If x and u both lie on some cycle in (the faulty) Q_n^k then this cycle is necessarily (x, y, u, v). Consequently, the value of $4n-5$ for the number of faulty edges in Q_n^k in the statement of Theorem 10 is optimal.

5 Conclusions

We end by presenting our conclusions and some open problems. Our main result is that Q_n^k with $4n-5$ faulty edges, but where every vertex is incident with at least 2 healthy edges, is bipancyclic, for every even k greater than or equal to 4. Ideally, we would like to prove that when Q_n^k has $4n-5$ faulty edges so that every vertex is incident with at least 2 healthy edges, Q_n^k is bipancyclic for all $k \geq 3$ (no matter what the parity of k). However, consider the proof of Theorem 10. There are a number of constructions within that proof that have drawbacks when k is odd. For example, we often extend a specific cycle C, of some (even) length m, by iteratively joining it to (even length) Hamiltonian cycles in some k-ary $(n-1)$-cubes contained within Q_n^k so as to obtain healthy cycles of all even lengths from m up to k^n in Q_n^k. When k is odd, this technique cannot be applied in such an elementary way as the Hamiltonian cycles of the k-ary $(n-1)$-cubes have odd length. Alternatively, extending our cycle C with healthy cycles of (even) length $k^{n-1}-1$ leaves us with a vertex in each such k-ary $(n-1)$-cube not contained within the resulting cycle. Also, by the same token, when we extend a healthy cycle using healthy $T_{s,t}$'s (as in the proof of Theorem 10), because k is odd we find that we have vertices not appearing on our cycles. It would appear that a significant amount of extra work has to be done (and possibly new techniques established) if one wishes to prove that Q_n^k is bipancyclic when k is odd. The same can be said as regards proving that Q_n^k is k-pancyclic when k is odd.

As remarked at the end of the previous section, Theorem 10 is optimal. The argument for optimality is that used in [2] to rule out Hamiltonian cycles in certain configurations of more that $4n-5$ faults in Q_n^k. It would be interesting to know in a situation where there are more than $4n-5$ faulty edges in Q_n^k (and Q_n^k still satisfies our conditional fault assumption), whether there is an upper bound of the form $k^n - m$ so that when $k \geq 4$ is even, there are healthy cycles of all lengths from 4 up to $k^n - m$ (of course, we would prefer that m is constant or at least a very slow growing function of n and possibly k). From the constructions in [2], the smallest m can be is 2.

Related to pancyclicity and bipancyclicity are two concepts. A graph G on n vertices is *panconnected* (resp. *bipanconnected*) if for any two distinct vertices u and v of G, there is a path of every length l for which $dist(u, v) \leq l \leq n$ (resp. for which $dist(u, v) \leq l \leq n$ and l and $dist(u, v)$ have the same parity), where $dist(u, v)$, for 2 vertices u and v of a graph, is the length of a shortest path joining u and v. Both panconnectivity and edge-pancyclicity (resp. bipanconnectivity and edge-bipancyclicity) imply pancyclicity (resp. bipancyclicity). It was proven in [24] that when $k \geq 3$ and $n \geq 2$, Q_n^k is bipannconnected and edge-bipancyclic. It would be interesting to know as to whether the main result of this paper can be extended to encompass bipannconnectivity or edge-bipancyclicity. However, we note that in order to prove such extensions, we will need radically different techniques to those employed in the proof of Theorem 10 which are decidedly non-uniform.

Finally, we mention the study of pancyclicity in arbitrary graphs which has a long history. In [5], Bondy made the following 'meta-conjecture': Almost any non-trivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic (there may be a simple family of exceptional graphs). The classical result giving such a condition is Dirac's Theorem [12] that says that every graph on $n \geq 3$ vertices that has minimum degree at least $\frac{n}{2}$ is Hamiltonian, and which was extended by Bondy [4] who showed that the same assumptions imply that a graph is either $K_{\frac{n}{2},\frac{n}{2}}$ or pancyclic. Other conditions include: if the connectivity of a graph G is no less than the independence number of G and G is triangle-free then G contains a cycle of every length from 4 up to n unless G is a cycle of length 5 or $G = K_{k,k}$, for some k [21] (Erdös [9] had shown that if a graph is such that its connectivity is no less than its independence number then the graph is Hamiltonian); and if G is a Hamiltonian graph with minimum degree at least 600 times the independence number of G then G is pancyclic [20]. Of course, results such as these are of no use to us when dealing with k-ary n-cubes but it would be interesting to study which conditions on an arbitrary (Hamiltonian) bipartite graph force the graph to be bipancyclic.

References

[1] Y.A. Ashir and I.A. Stewart, On embedding cycles in k-ary n-cubes, *Parallel Processing Letters* 7 (1997) 49–55.

[2] Y.A. Ashir and I.A. Stewart, Fault-tolerant embeddings of Hamiltonian circuits in k-ary n-cubes, *SIAM Journal on Discrete Mathematics* 15 (2002) 317–328.

[3] S. Bettayeb, On the k-ary hypercube, *Theoretical Computer Science* 140 (1995) 333–339.

[4] J.A. Bondy, Pancyclic graphs I, *Journal of Combinatorial Theory, Series B* 11 (1971) 80–84.

[5] J.A. Bondy, Pancyclic graphs - recent results, in: *Infinite and Finite Sets* (A. Hajnal, R. Rado, V.T. S'os, eds.), Colloquia Mathematica Societatis J'anos Bolyai 10, North-Holland, Amsterdam (1975) 181–187.

[6] S. Borkar, R. Cohn, G. Cox, S. Gleason, T. Gross, H. T. Kung, M. Lam, B. Moore, C. Peterson, J. Pieper, L. Rankin, P. S. Tseng, J. Sutton, J. Urbanski, and J. Webb, iWarp: an integrated solution to high-speed parallel computing, *Proc. IEEE Supercomputing* 88, IEEE Computer Society Press (1988) 330–339.

[7] B. Bose, B. Broeg, Y. Kwon and Y. Ashir, Lee distance and topological properties of k-ary n-cubes, *IEEE Transations on Computers* 44 (1995) 1021–1030.

[8] M.Y. Chang and S.-J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, *SIAM Journal on Discrete Mathematics* 4 (1991) 511–527.

[9] V. Chvátal and P. Erdös, A note on Hamiltonian circuits, *Discrete Mathematics* 2 (1972) 111–113.

[10] Z. Cvetanovic, Performance analysis of the Alpha 21364-based HP GS1280 multiprocessor, *Proc. of* 30*th Ann. Int. Symp. on Computer Architecture*, ACM Press (2003) 218–228.

[11] W. Dally, Performance analysis of k-ary n-cube interconnection networks, *IEEE Transaction on Computers* 39 (1990) 775–785.

[12] G.A. Dirac, Some theorems on abstract graphs, *Proceedings of the London Mathematical Society* 2 (1952) 69–81.

[13] J.-S. Fu, Conditional fault-tolerant Hamiltonicity of star graphs, *Parallel Computing* 33 (2007) 488–496.

[14] J.-S. Fu, Fault-free Hamiltonian cycles in twisted cubes with conditional link faults, *Theoretical Computer Science* 407 (2008) 318–329.

[15] S.-Y. Hsieh and C.-W. Lee, Conditional edge-fault Hamiltonicity of matching composition networks, *IEEE Transactions on Parallel and Distributed Systems* 20 (2009) 581–592.

[16] S.-Y. Hsieh, T.-J. Lin and H.-L. Huang, Panconnectivity and edge-pancyclicity of 3-ary n-cubes, *Journal of Supercomputing* 42 (2007) 225–233.

[17] S.-Y. Hsieh and C.-D. Wu, Conditional edge-fault-tolerant Hamiltonian cycle embedding of star graphs, *Proc.* 13*th Int. Conf. on Parallel and Distributed Systems* (*ICPADS* 07), IEEE Computer Society Press (2007) 1–8.

[18] S.-Y. Hsieh and C.-Y. Wu, Edge-fault-tolerant Hamiltonicity of locally twisted cubes under conditional edge faults, *Journal of Combinatorial Optimization* 19 (2010) 16–30.

[19] H.-S. Hung, J.-S. Fu and G.-H. Chen, Fault-free Hamiltonian cycles in crossed cubes with conditional link faults, *Information Sciences* 177 (2007) 5664–5674.

[20] P. Keevash and B. Sudakov, Pancyclicity of Hamiltonian and highly connected graphs, *Journal of Combinatorial Theory, Series B* 100 (2010) 456–467.

[21] D. Lou, The Chvátal–Erdös condition for cycles in triangle-free graphs, *Discrete Mathematics* 152 (1996) 253–257.

[22] S.L. Scott and G.M. Thorson, The Cray T3E network: adaptive routing in a high performance 3D torus, *Proc. Symp. High Performance Interconnects* (*Hot Interconnects IV*) (1996) 147–156.

[23] I.A. Stewart, Distributed algorithms for building Hamiltonian cycles in k-ary n-cubes and hypercubes with faulty links, *Journal of Interconnection Networks* 8 (2007) 253–284.

[24] I.A. Stewart and Y. Xiang, Bipanconnectivity and bipancyclicity in k-ary n-cubes, *IEEE Transactions on Parallel and Distributed Systems* 20 (2009) 25–33.

[25] C.-H. Tsai, Linear array and ring embeddings in conditional faulty hypercubes, *Theoretical Computer Science* 314 (2004) 431–443.

[26] P.-Y. Tsai, J.-S. Fu and G.-H. Chen, Embedding Hamiltonian cycles in alternating group graphs under conditional fault model, *Information Sciences* 179 (2009) 851–857.

[27] D. Wang, T. An, M. Pan, K. Wang and S. Qu, Hamiltonian-like properties of k-ary n-cubes, *Proc. of* 6*th Int. Conf. on Parallel and Distributed Computing, Applications and Technologies* (*PDCAT* 05), IEEE Computer Society Press (2005) 1002–1007.

[28] H. Wang, L.-S. Peh and S. Malik, A technology-aware and energy-oriented topology exploration for on-chip networks, *Proc. of Conf. on Design, Automation and Test in Europe*, IEEE Computer Society Press (2005) 1238–1243.